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ORIGINAL PAPER

Existence results for inequality problems on various subsets of Banach spaces and applications

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Abstract Several existence results for a class of perturbed inequality problems on various subsets of Banach spaces are proved. A concrete application to a problem from Nonsmooth Mechanics illustrates is given.

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1 Introduction

The theory of the existence of solutions to variational inequalities (introduced around 1965 with the pioneering works of Fichera [5], Lions and Stampacchia [10]) is a well-developed theory in mathematics which is closely connected with the convexity of the energy functionals involved and is based on monotonicity arguments. On the contrary, the hemivariational inequalities (introduced by Panagiotopoulos, starting from the notion of generalized gradient of Clarke) are much more general in the sense that they are not equivalent to minimum problems but give rise to substationarity problems. The reader can find in [1, 2, 6–8, 11–15] a rich bibliography in this field and in [13, 14] a lot of applications to certain Mechanics, Engineering, and Economics problems (e.g., in the buckling theory of Kirchhoff or laminated von Kármán plate, in (non)convex semipermeability problems, in the theory of multilayered plates (delamination), in the theory of composite structures, in the theory of partial debonding of adhesive joints, in network flow problems, etc.).

This paper is dedicated to the memory of Professor P.D. Panagiotopoulos

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2 Hypotheses and preliminaries

Throughout this paper we will consider that Ω is a bounded subset of $\mathbb{R}^N$ $(N \geq 1)$, V is a Banach space, $T: V \to L^p\left(\Omega, \mathbb{R}^k\right)$ is a linear and continuous operator for some $1 \leq p < \infty$, $k \geq 1$, $A, B: K \to V^*$ are two operators (possibly nonlinear), K is a subset (possibly unbounded) of V, $G: V \to V$ is an operator with $G \in L(V)$, $j: \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a function defined for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^k$, f is an element of V^*, and $\Phi: V \to \mathbb{R} \cup \{+\infty\}$ is a convex, lower semicontinuous functional. We will denote $\widehat{u} = (T \circ G)(u)$, $\forall u \in V$ and $\langle \cdot, \cdot \rangle$ the duality application between V^* and V.

Our aim is to study the following inequality problem:

Find $u \in K$ such that, for every $v \in K$,

$$\langle (A+B)u - f, Gv - Gu \rangle + \Phi(v) - \Phi(u) + \int_\Omega j^0\left(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x)\right) \mathrm{d}x \geq 0 \tag{2.1}$$

(for the treatment of the variational inequality part of (2.1) see, e.g. [16, 17]).

In what follows, we impose on j the following conditions:

(j') For all $\xi \in \mathbb{R}^k$ the function $\Omega \ni x \to j(x, \xi)$ (as a function of the variable x) is measurable on Ω.

(j") For almost all $x \in \Omega$ the function $\mathbb{R}^k \ni \xi \to j(x, \xi)$ (as a function of the variable ξ) is locally Lipschitz on $\mathbb{R}^k$.

The properties (j') and (j") ensure that j belongs to the class of functions having the Carathéodory property (see [3]). Notice that in the original definition of the Carathéodory property the condition (j") requires the corresponding function to be only continuous on $\mathbb{R}^N$. However, for our purposes, the continuity must be strengthened and thus here the property "locally Lipschitz" is assumed to hold. This fact permits us to define for almost all $x \in \Omega$ the differential of Clarke according to the formula

$$j^0(x, \xi; \eta) = \limsup_{h \to 0,\, \lambda \to 0_+} \frac{j(x, \xi + h + \lambda\eta) - j(x, \xi + h)}{\lambda}, \quad \xi,\ \eta \in \mathbb{R}^N$$

and the generalized gradient of Clarke

$$\partial j(x, \xi) = \left\{\eta \in \mathbb{R}^N,\ j^0(x, \xi; \eta) \geq \eta \cdot \gamma,\ \forall \gamma \in \mathbb{R}^N\right\}$$

for almost $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

Furthermore, in our study we will admit that j fulfills the following growth condition:

(j) there exist $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R})$ and $h_2 \in L^\infty(\Omega, \mathbb{R})$ such that, for almost all $x \in \Omega$, for all $\xi \in \mathbb{R}^k$, and for all $z \in \partial j(x, \xi)$, we have

$$|z| \leq h_1(x) + h_2(x)\, |\xi|^{p-1}.$$

(By the symbols "$|\cdot|$" and "$\cdot$" we denote the norm and the inner product in the Euclidean space $\mathbb{R}^N$, respectively.)

As in [15] one can easily deduce the following useful result.

Lemma 2.1 (a) *Suppose that hypothesis* (j) *is fulfilled and* V_1, V_2 *are nonempty subsets of* V. *Then the mapping* $V_1 \times V_2 \ni (u, v) \to \int_\Omega j^0(x, \widehat{u}(x); \widehat{v}(x))\, \mathrm{d}x$ *is upper semicontinuous.*

(b) *If, in addition, T is compact, the above mapping is weakly upper semicontinuous.*

In what follows we need to recall the following definitions (see, e.g. [16, 17]).

Definition 2.1 The operator $A\colon K \to V^*$ is called w^*-**demicontinuous** if for any sequence $(u_n)_n$ converging to u, the sequence $(Au_n)_n$ converges to Au in the w^*-topology of V^*.

Definition 2.2 The operator $A\colon K \to V^*$ is called **continuous on finite dimensional subspaces** of K if for any finite dimensional space $F \subset V$, which intersects K, the operator $A|_{K\cap F}$ is w^*-demicontinuous, that is $(Au_n)_n$ converges weakly to Au in V^*, for each sequence $(u_n)_n \subset K \cap F$ which converges to u.

In Sect. 3, we will present two existence results to problem (2.1) in the case when K is bounded and Sect. 4 contains existence results to problem (2.1) in the case when K is not necessarily bounded. Finally, in the last section, a concrete application to a problem from Nonsmooth Mechanics is given.

3 Existence results on bounded sets

In this section, we will present two existence results to problem (2.1) in the case when K is bounded.

Theorem 3.1 *Suppose that V is finite dimensional, K is nonempty, convex, and compact, and A, B are continuous and linear operators. Then, there exists $u \in K$ such that, for every $v \in K$* (2.1) *is fulfilled.*

Proof The proof follows by using a standard technique of Knaster–Kuratowski–Mazurkiewicz (KKM, in short) type (see [4, 9]).

Suppose, by contradiction, that for every $u \in K$, there exists $v = v\,(u) \in K$ such that

$$\langle (A+B)\,u - f, Gv - Gu\rangle + \Phi\,(v) - \Phi\,(u) + \int\limits_{\Omega} j^0\,(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x))\,\mathrm{d}x < 0. \tag{3.1}$$

For each $v \in K$ set

$$S\,(v) = \left\{ u \in K,\ \ \langle (A+B)\,u - f, Gv - Gu\rangle + \Phi\,(v) - \Phi\,(u) + \int\limits_{\Omega} j^0(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x))\mathrm{d}x < 0 \right\}.$$

Since A, B, and G are continuous operators, by using Lemma 2.1, we get that $S\,(v)$ is an open set, for every $v \in K$.

Let $u \in K$ be arbitrarily fixed. Then there exists $v \in K$ fulfilling relation (2.1). So, $u \in S\,(v)$. Hence, $K \subseteq \bigcup_{v\in K} S\,(v)$. Since K is compact, there exists $\{v_1, \ldots, v_m\} \subseteq K$ such that $K \subseteq \bigcup_{i=1}^m S\,(v_i)$, $m \in \mathbb{N}\setminus\{0\}$.

Let us define $d_i(u) := \text{dist}\,(u, K\backslash S(v_i))$, $\forall i \in \overline{1,m}$ (i.e. the distance from u to $K\backslash S(v_i)$). It is easily seen that d_i is a Lipschitz map, $d_i = 0$ on $K\backslash S(v_i)$, $\forall i \in \overline{1,m}$, and the functionals $\mu_i(u) := d_i(u) / \left(\sum_{j=1}^{m} d_j(u)\right)$, $\forall i \in \overline{1,m}$ define a partition of the unity related to the covering $\{d_1, \ldots, d_m\}$. Furthermore, the mapping $F(u) := \sum_{i=1}^{m} \mu_i(u)\, v_i$ is continuous and, since the convexity of K, F maps K into itself.

By applying the Brouwer's fixed point theorem, F admits a fixed point in the convex hull of $\{v_1, \ldots, v_m\}$, say u_0 and let us define

$$H(u) := \langle (A+B)\,u - f, Gu - GFu\rangle + \Phi(u) - \Phi(Fu) + \int_\Omega j^0\left(x, \widehat{u}(x); \widehat{u}(x) - \widehat{Fu}(x)\right) \mathrm{d}x.$$

Then, easy estimates lead us to

$$H(u) \geq \sum_{i=1}^{m} \mu_i(u) \{\langle (A+B)\,u_i - f, Gu - Gv_i\rangle + \Phi(u) - \Phi(v_i)\} + \int_\Omega j^0\left(x, \widehat{u}(x); \sum_{i=1}^{m} \mu_i(u)\,(\widehat{u}(x) - \widehat{v_i}(x))\right) \mathrm{d}x.$$

For each $u \in K$, there are two exclusive and exhaustive possibilities. If $u \notin S(v_i)$, $\forall i \in \overline{1,m}$, then $\mu_i(u) = 0$. If $u \in S(v_j)$, with $j \in \overline{1,m}$, we have $\mu_j(u) > 0$.

Hence we may conclude that $H(u) > 0$, $\forall u \in K$, which contradicts the fact that $H(u_0) = 0$.

The proof of Theorem 3.1 is now complete. □

Theorem 3.2 *Suppose that V is infinite dimensional, K is nonempty, convex, and compact, G is finite dimensional operator* (*i.e.* $\dim \operatorname{Im} G < \infty$), *and the following conditions are fulfilled:*

(1) *$A + B$ is w^*-demicontinuous;*
(2) *The function $u \longmapsto \langle (A+B)\,u - f, Gu\rangle$ is lower semicontinuous.*

Then, there exists $u \in K$ such that, for every $v \in K$ (2.1) *is fulfilled.*

Proof For $v \in K$ set

$$K(v) = \left\{ u \in K,\ \langle (A+B)\,u - f, Gv - Gu\rangle + \Phi(v) - \Phi(u) + \int_\Omega j^0(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x))\mathrm{d}x \geq 0 \right\}.$$

Since $v \in K(v)$, it follows that $K(v)$ is nonempty. We prove that $K(v)$ is closed. To this purpose, let us consider $(u_n)_{n\in\mathbb{N}} \subset K(v)$, $u_n \to u$. So,

$$\langle (A+B)\,u_n - f, Gv - Gu_n\rangle + \Phi(v) - \Phi(u_n) + \int_\Omega j^0\,(x,\, \widehat{u_n}(x); \widehat{v}(x) - \widehat{u_n}(x))\,\mathrm{d}x \geq 0 \tag{3.2}$$

for each $n \in \mathbb{N}$ and $u_n \rightharpoonup u$.

Hence, by hypothesis (2), relation (3.2), and the lower semicontinuity of Φ, we get

$$
\begin{aligned}
0 &\leq \limsup_{n\to\infty} \langle (A+B)\, u_n - f, Gv\rangle - \liminf_{n\to\infty} \langle (A+B)\, u_n - f, Gu_n\rangle \\
&\quad + \Phi(v) - \liminf_{n\to\infty} \Phi(u_n) + \limsup_{n\to\infty} \int_\Omega j^0\left(x, \widehat{u_n}(x); \widehat{v}(x) - \widehat{u_n}(x)\right) \mathrm{d}x \\
&\leq \langle (A+B)\, u - f, Gv - Gu\rangle + \Phi(v) - \Phi(u) + \int_\Omega j^0\left(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x)\right) \mathrm{d}x.
\end{aligned}
$$

Therefore, $u \in K(v)$. The family $\{K(v),\ v \in K\}$ admits the finite intersection property. Indeed, we consider $\{v_1, \ldots, v_p\} \subset K$, $p \in \mathbb{N}\setminus\{0\}$, a finite family and let $M = \mathrm{Span}(\{v_1, \ldots, v_p\} \cup \mathrm{Im}\, G)$, $\dim M < \infty$. Let $i_M : M \to V$ be the canonical injection and $i_M^* : V^* \to M^*$ its adjoint. Let C be the operator $i_M^*(A+B)\, i_M : K\cap M \to M^*$. Then, C is continuous. Indeed, if $(u_n)_{n\in\mathbb{N}} \subset K \cap M$, $u_n \to u$ in V, we have:

$$
\langle Cu_n, v\rangle = \left\langle i_M^*(A+B)\, i_M u_n, v\right\rangle = \langle (A+B)\, u_n, v\rangle \to \langle (A+B)\, u, v\rangle = \langle Cu, v\rangle .
$$

Hence, $Cu_n \rightharpoonup Cu$ in the topology $\sigma(M^*, M)$. Since M is a finite dimensional space, it follows that $Cu_n \to Cu$, in M^*.

By Theorem 3.1, there exists $u \in K \cap M$, such that for each $v \in K \cap M$,

$$
\langle Cu - f, Gv - Gu\rangle + \Phi(v) - \Phi(u) + \int_\Omega j^0(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x))\mathrm{d}x \geq 0.
$$

In particular, for each $i \in \overline{1,p}$, one has $u \in K(v_i)$.

Hence, there exists $u \in \bigcap_{v\in K} K(v)$, i.e. there exists $u \in K$, such that for every $v \in K$, relation (2.1) is fulfilled.

The proof of Theorem 3.2 is now complete. □

Remark 3.1 It is readily seen that Theorem 3.2 works as well as in the case when K is weakly compact set and T is compact operator.

4 Existence results on unbounded sets

In this section, we will present two existence results to problem (2.1) in the case when K is not necessarily bounded.

Theorem 4.1 *Suppose that V is infinite dimensional, K is nonempty, convex, and closed, G is finite dimensional operator, Φ is weakly sequentially lower semicontinuous, T is compact, there exists $u^* \in K$, such that*

$$
\liminf_{u\in K,\ \|u\|\to\infty} \left\{ \left\langle (A+B)\, u - f, Gu - Gu^*\right\rangle - \Phi\left(u^*\right) + \Phi(u) - \int_\Omega j^0\left(x, \widehat{u}(x); -\widehat{u}(x) + \widehat{u^*}(x)\right) \mathrm{d}x \right\} > 0, \tag{4.1}
$$

hypothesis (1) *of Theorem* 3.2 *and the following are fulfilled:*

(a) *If* $u_n \rightharpoonup u$, *then* $\liminf_{n\to\infty} \langle (A+B)\,u_n,\, v\rangle \le \langle (A+B)\,u, v\rangle$, *for all* $v \in K$;
(b) *The function* $u \longmapsto \langle (A+B)\,u - f, Gu\rangle$ *is weakly sequentially lower semicontinuous.*

Then, there exists $u \in K$ *such that, for every* $v \in K$ *(2.1) is fulfilled.*

Proof Let m be a positive integer, such that $u^* \in \overline{B_m}(0)$. Since V is reflexive, $\overline{B_n}(0)$ is nonempty, convex, and weakly compact for all $n \ge m$. So, by Theorem 3.2 and taking into account Remark 3.1, for each $n \ge m$, there exists $u_n \in \overline{B_n}(0)$, such that

$$\langle (A+B)\,u_n - f, Gv - Gu_n\rangle + \Phi\,(v) - \Phi\,(u_n) + \int_\Omega j^0\,(x, \widehat{u_n}\,(x)\,; \widehat{v}\,(x) - \widehat{u_n}\,(x))\,\mathrm{d}x \ge 0 \tag{4.2}$$

for all $v \in \overline{B_n}(0)$. In particular,

$$\begin{aligned}&\left\langle (A+B)\,u_n - f, Gu^* - Gu_n\right\rangle + \Phi\left(u^*\right) - \Phi\,(u_n)\\ &\quad + \int_\Omega j^0\left(x, \widehat{u_n}\,(x)\,; \widehat{u^*}\,(x) - \widehat{u_n}\,(x)\right)\mathrm{d}x \ge 0\end{aligned} \tag{4.3}$$

for all $n \ge m$.

Remark that $(u_n)_{n\in\mathbb{N}}$ is bounded, since, otherwise, by (4.1) and (4.3) we would obtain (by passing eventually to subsequences)

$$\begin{aligned}0 &< \liminf_{n\to\infty}\Bigg\{\left\langle (A+B)\,u_n - f, Gu_n - Gu^*\right\rangle - \Phi\left(u^*\right) + \Phi\,(u_n)\\ &\quad - \int_\Omega j^0\left(x, \widehat{u_n}\,(x)\,; -\widehat{u_n}\,(x) + \widehat{u^*}\,(x)\right)\mathrm{d}x\Bigg\}\\ &\le 0.\end{aligned}$$

Therefore, since V is reflexive, $(u_n)_{n\in\mathbb{N}}$ (or one of its subsequences) converges weakly to some $u \in K$.

Consider $v \in K$ arbitrarily fixed and let $q = q\,(v) \ge m$ $(q \in \mathbb{N})$, such that $v \in \overline{B_q}\,(0)$. We get from (4.2)

$$\langle (A+B)\,u_n - f, Gv - Gu_n\rangle + \Phi\,(v) - \Phi\,(u_n) + \int_\Omega j^0\,(x, \widehat{u_n}\,(x)\,; \widehat{v}\,(x) - \widehat{u_n}\,(x))\,\mathrm{d}x \ge 0 \tag{4.4}$$

for all $n \geq q$. Then, taking into account hypotheses (a) and (b), relation (4.4), and the weakly sequential lower semicontinuity of Φ, we deduce successively

$$\begin{aligned}\langle (A+B)\,u - f, Gu\rangle &\leq \liminf_{n\to\infty} \langle (A+B)\,u_n - f, Gu_n\rangle \\ &\leq \liminf_{n\to\infty} \Bigg\{ \langle (A+B)\,u_n - f, Gv\rangle + \Phi\,(v) - \Phi\,(u_n) \\ &\quad + \int_\Omega j^0\,(x, \widehat{u_n}\,(x)\,; \widehat{v}\,(x) - \widehat{u_n}\,(x))\,\mathrm{d}x \Bigg\} \\ &\leq \liminf_{n\to\infty} \langle (A+B)\,u_n - f, Gv\rangle + \Phi\,(v) - \liminf_{n\to\infty} \Phi\,(u_n) \\ &\quad + \limsup_{n\to\infty} \int_\Omega j^0\,(x, \widehat{u_n}\,(x)\,; \widehat{v}\,(x) - \widehat{u_n}\,(x))\,\mathrm{d}x \\ &\leq \langle (A+B)\,u - f, Gv\rangle + \Phi\,(v) - \Phi\,(u) \\ &\quad + \int_\Omega j^0\,(x, \widehat{u}\,(x)\,; \widehat{v}\,(x) - \widehat{u}\,(x))\,\mathrm{d}x.\end{aligned}$$

This ends the proof of Theorem 4.1. □

Theorem 4.2 *Suppose that V is infinite dimensional, K is nonempty, convex, and closed, Φ is weakly sequentially lower semicontinuous, T is compact, there exists $u^* \in K$ such that relation* (4.1) *and the following conditions are fulfilled:*

(1) *$A+B$ is w^*-demicontinuous;*
(2) *If $u_n \rightharpoonup u$ then $\liminf_{n\to\infty} \langle Au_n - f, Gv - Gu_n\rangle \leq \langle Au - f, Gv - Gu\rangle$, for all $v \in K$;*
(3) *The function $u \longmapsto \langle Bu - f, Gv - Gu\rangle$ is weakly sequentially upper semicontinuous.*

Then, there exists $u \in K$ such that, for every $v \in K$ (2.1) *is fulfilled.*

In order to prove Theorem 4.2 we need the following Lemma.

Lemma 4.1 *Suppose that V is infinite dimensional, K is nonempty, convex, and weakly compact, Φ is weakly sequentially lower semicontinuous, T is compact, and hypotheses* (1)–(3) *of Theorem* 4.2 *are fulfilled. Then, there exists $u \in K$ such that, for every $v \in K$* (2.1) *is fulfilled.*

Proof of Lemma 4.1 Set $\Lambda = \{U \subset V,\ U \text{ finite dimensional}\}$, ordered by inclusion.

For $U \in \Lambda$, let i_U be the injection of U into V and i_U^* be the adjoint of i_U. Then, the operator $i_U^*(A+B)i_U : K\cap U \to U^*$ is continuous from the strong topology of $K \cap U$ to the weak topology of U^*. Since $K \cap U$ is convex, compact, and finite dimensional, by applying Theorem 3.1, there exists $u_U \in K \cap U$, such that for all $v \in K \cap U$,

$$\begin{aligned}&\left\langle i_U^*(A+B)i_U u_U - f, Gv - Gu_U\right\rangle + \Phi\,(v) - \Phi\,(u_U) \\ &\quad + \int_\Omega j^0\,(x, \widehat{u_U}\,(x)\,; \widehat{v}\,(x) - \widehat{u_U}\,(x))\,\mathrm{d}x \geq 0.\end{aligned}$$

Hence, there exists $u_U \in K \cap U$ such that for all $v \in K \cap U$,

$$\begin{aligned}&\langle (A+B)u_U - f, Gv - Gu_U\rangle + \Phi\,(v) - \Phi\,(u_U) \\ &\quad + \int_\Omega j^0\,(x, \widehat{u_U}\,(x)\,; \widehat{v}\,(x) - \widehat{u_U}\,(x))\,\mathrm{d}x \geq 0. \qquad (4.5)\end{aligned}$$

For $U \in \Lambda$, denote $K_U := \{u_V,\ U \subset V \in \Lambda\}$ and let $\overline{K_U}$ be the weak closure of K_U. Then, since the family $\left\{\overline{K_U},\, U \in \Lambda\right\}$ has the finite intersection property and K is weakly compact, $\bigcap_{U\in\Lambda} \overline{K_U} \neq \emptyset$. So, let $u \in \bigcap_{U\in\Lambda} \overline{K_U}$ and consider $v \in K$ arbitrarily fixed. Let $U \in \Lambda$, such that $\{u, v\} \subset U$. Since V is reflexive and $u \in \bigcap_{U\in\Lambda} \overline{K_U}$, there exists $(U_n)_n$, $U_n \supset U$, such that $u_n := u_{U_n} \rightharpoonup u$ and $u_n \in K_{U_n}$.

From (4.5) we deduce

$$\langle (A+B)u_n - f, Gv - Gu_n\rangle + \Phi(v) - \Phi(u_n) + \int_\Omega j^0\left(x, \widehat{u_n}(x); \widehat{v}(x) - \widehat{u_n}(x)\right) \mathrm{d}x \geq 0. \tag{4.6}$$

Hence, by hypotheses (2), (3) and relation (4.6), we find

$$\begin{aligned}
&\langle (A+B)u - f, Gv - Gu\rangle + \Phi(v) - \Phi(u) + \int_\Omega j^0\left(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x)\right) \mathrm{d}x \\
&\quad\geq \liminf_{n\to\infty} \langle Au_n - f, Gv - Gu_n\rangle + \limsup_{n\to\infty} \langle Bu_n - f, Gv - Gu_n\rangle \\
&\qquad +\Phi(v) - \liminf_{n\to\infty} \Phi(u_n) + \limsup_{n\to\infty} \int_\Omega j^0\left(x, \widehat{u_n}(x); \widehat{v}(x) - \widehat{u_n}(x)\right) \mathrm{d}x \\
&\quad\geq \liminf_{n\to\infty} \left\{ \langle (A+B)u_n - f, Gv - Gu_n\rangle + \Phi(v) - \Phi(u_n) \right. \\
&\qquad \left. + \int_\Omega j^0\left(x, Tu_n(x); Tv(x) - Tu_n(x)\right) \mathrm{d}x \right\} \\
&\quad\geq 0.
\end{aligned}$$

So, there exists $u \in K$, such that for every $v \in K$ (2.1) is fulfilled. □

Proof of Theorem 4.2 Let m be a positive integer, such that $u^* \in \overline{B_m}(0)$. Since V is reflexive, $\overline{B_n}(0)$ is nonempty, convex, and weakly compact for each $n \geq m$. Thus, by applying Lemma 4.1, for each $n \geq m$, there exists $u_n \in \overline{B_n}(0)$ such that, for every $w \in \overline{B_n}(0)$,

$$\begin{aligned}
&\langle (A+B)u_n - f, Gw - Gu_n\rangle + \Phi(w) - \Phi(u_n) \\
&\quad + \int_\Omega j^0\left(x, \widehat{u_n}(x); \widehat{w}(x) - \widehat{u_n}(x)\right) \mathrm{d}x \geq 0.
\end{aligned} \tag{4.7}$$

In particular,

$$\begin{aligned}
&\left\langle (A+B)u_n - f, Gu^* - Gu_n\right\rangle + \Phi\left(u^*\right) - \Phi(u_n) \\
&\quad + \int_\Omega j^0\left(x, \widehat{u_n}(x); \widehat{u^*}(x) - \widehat{u_n}(x)\right) \mathrm{d}x \geq 0.
\end{aligned} \tag{4.8}$$

Remark that $(u_n)_{n\in\mathbb{N}}$ is bounded, since, otherwise, by (4.1) and (4.3) we would obtain (by passing eventually to subsequences)

$$
\begin{aligned}
0 < \liminf_{n\to\infty} & \Big\{ \left\langle (A+B)u_n - f, Gu_n - Gu^* \right\rangle - \Phi\left(u^*\right) + \Phi\left(u_n\right) \\
& - \int_\Omega j^0\left(x, \widehat{u_n}(x); \widehat{u^*}(x) - \widehat{u_n}(x)\right) \mathrm{d}x \Big\} \\
\leq 0. &
\end{aligned}
$$

Therefore, since V is reflexive, $(u_n)_{n\in\mathbb{N}}$ (or one of its subsequences) converges weakly to some $u \in K$.

Let $n_0 \in \mathbb{N}$, such that $n_0 \geq m$, and $u \in \overline{B_{n_0}}(0)$, and consider $w \in \overline{B_{n_0}}(0)$ arbitrarily fixed. Then, from hypotheses (2), (3), relation (4.7), and the weak sequential lower semicontinuity of Φ, it follows that

$$
\begin{aligned}
0 \leq & \liminf_{n\to\infty} \langle Au_n - f, Gw - Gu_n \rangle + \limsup_{n\to\infty} \langle Bu_n - f, Gw - Gu_n \rangle \\
& + \Phi(w) - \liminf_{n\to\infty} \Phi(u_n) + \limsup_{n\to\infty} \int_\Omega j^0(x, \widehat{u_n}(x); \widehat{w}(x) - \widehat{u_n}(x))\, \mathrm{d}x \\
\leq & \langle (A+B)u - f, Gw - Gu \rangle + \Phi(w) - \Phi(u) \\
& + \int_\Omega j^0(x, \widehat{u}(x); \widehat{w}(x) - \widehat{u}(x))\, \mathrm{d}x. \qquad (4.9)
\end{aligned}
$$

Let $v \in K$ be arbitrarily fixed. We choose $\lambda > 0$ small enough such that

$$
w_\lambda := \lambda v + (1-\lambda)u \in \overline{B_{n_0}}(0).
$$

By substituting w_λ into (4.9), we obtain

$$
0 \leq \langle (A+B)u - f, Gv - Gu \rangle + \Phi(v) - \Phi(u) + \int_\Omega j^0(x, \widehat{u}(x); \widehat{v}(x) - \widehat{u}(x))\, \mathrm{d}x
$$

and proof of Theorem 4.2 is now complete. □

5 Application

In this section, we will give an application of Theorem 3.2 to a problem from Nonsmooth Mechanics, regarding an adhesively supported elastic plate between two rigid suppots. Let us consider a Kirchoff plate. The elastic plate is referred to a right-handed orthogonal Cartesian coordinate system $Ox_1x_2x_3$. The plate is supposed to have constant thickness h_1. We also assume that the middle surface of the plate coincides with the Ox_1x_2-plane. Let Ω be an open, bounded and connected subset of $\mathbb{R}^2$ with $C^{0,1}$ boundary Γ. The domain Ω is occupied by the plate in its undeformed state. On $\Omega' \subset \Omega$ $\left(\Omega' \text{ is such that } \overline{\Omega'} \cap \Gamma = \emptyset\right)$ the plate is bounded to a support through an adhesive material. We denote by $\zeta(x)$ the defection of the point $x = (x_1, x_2, x_3)$ and by $g = (0, 0, g_3)$, $g_3 = g_3(x)$ (hereafter called g for simplicity) the distributed load of the considered plate per unit area of the middle surface. Concerning the laws for the adhesive forces and the formulation of the problems we refer the reader to [14]. We make the additional assumption that the displacements of the plate are prevented by some rigid supports. Thus we may put as an additional assumption the following one:

$$v \in K, \tag{5.1}$$

where K is a convex, closed, bounded subset of the displacement space. One could have, e.g., that $a_0 \leq v \leq b_0$, etc.

We assume that any type of boundary conditions may hold on Γ. Admit that the plate is coercive. Thus the whole space $H^2(\Omega)$ is the kinematically admissible set of the plate. If one takes now into account the relation (5.1), then $v \in K \subset H^2(\Omega)$, and our problem becomes:

Find $u \in K$ such that, for every $v \in K$,

$$a(u, v-u) + \int_{\Omega'} j^0(x, u(x); v(x) - u(x))\,\mathrm{d}x \geq (g, v-u). \tag{5.2}$$

Here $a(\cdot,\cdot)$ is the elastic energy of the Kirchhoff plate, i.e.,

$$a(u,v) = k \int_{\Omega} \left[(1-\nu)\, u_{,\alpha\beta}(x) v_{,\alpha\beta}(x) + \nu \Delta u(x) \Delta v(x)\right] \mathrm{d}x, \quad \alpha,\ \beta = 1,2, \tag{5.3}$$

where $k = \frac{Eh^3}{12(1-\nu^2)}$ is the bending rigidity of the plate with E and ν the modulus of elasticity and the Poisson ratio, respectively, and h is its thickness. Moreover, j is the binding energy of the adhesive which is a locally Lipschitz function on $H^2(\Omega)$ and $g \in L^2(\Omega)$, denotes the external fires. Furthermore, if j fulfils the growth condition (j) then, by taking into account that $a(\cdot,\cdot)$ appearing in (5.3) is continuous, we deduce, by Theorem 3.2 (with $\langle Au, v\rangle = a(u,v) + \int_{\Omega} fv\,\mathrm{d}x$, $B(v) = 0$, $T(v) = G(v) = v$, $\Phi(v) = \int_{\Omega}(-g)v\,\mathrm{d}x$, $\forall u,\ v$, and $f \in V^*$ is a prescribed element), the existence of a solution to (5.2).

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